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Explicit solitary and periodic solutions for optical cascading

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Abstract. Many types of periodic solutions to the pair of nonlinear partial differential equations describing optical cascading are obtained in terms of Jacobian elliptic functions. The choice of appropriate forms for periodic solutions is based on the known solitary-wave solutions of the system. By a straightforward procedure for determining coefficients, it is possible to construct periodic analogues of some classes of solitary wave solutions.

Key words: optical cascading, exact solutions, Jacobian elliptic functions, spatial vector solitons.

1. Introduction

Localized solutions (vector solitons) of coupled nonlinear Schrödinger (NLS) equations are of interest in nonlinear optics and telecommunications, since many waveguides are capable of supporting more than one mode with closely matched phase and group speeds. Recently, a number of proposals for exploiting coupled pulses or beams in compact devices using crystals with significant quadratic nonlinearity have been proposed, following experimental demonstration of 'spatial vector solitons' using the cascading process in which a beam at a fundamental frequency couples resonantly to one at the second harmonic frequency.

A number of explicit solutions to the quadratically coupled 'cascading' p.d.e.s have been found (Karamzin and Sukhorukov [1], Hayata and Koshiba [2], Werner and Drumond [3, 4], Menyuk, Shiek and Torner [5], Parker [6]). Those found in [1–4] are bright-bright solitons, while some bright-dark solitons appear in [5] and reference [6] includes many possibilities in which both modes have constant (nonzero) amplitude at a distance from the soliton.

Just as coupled NLS systems have solutions with periodic envelopes (see *e.g.* [7, 8]) described by Jacobian elliptic functions, it transpires that the cascading equations also possess solutions involving elliptic functions [9]. This paper details all such solutions found by an exhaustive search procedure and relates them to the known vector soliton solutions. Typical profiles are illustrated, showing how, as the period increases, the solutions become like periodically spaced solitary wave solutions.

Since the governing Equations (3) for optical cascading may also describe resonant interactions between a fundamental and a second-harmonic signal in other branches of physical science, the solutions described here should have broad applicability. They certainly are only a subset of all possible travelling-wave or stationary solutions (described by bounded solutions of Equations (6) and (7)). While the present analysis does not address the question of stability, it does indicate the rich variety of possible waveforms. Each provides a starting point for a numerical search for stable waveforms using parameter continuation.

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2. Formulation and solitary solutions

In a planar waveguide having optical properties depending only on the Cartesian coordinate x_3, the electric field of a linearized guided mode may be written as

$$\boldsymbol{E} = \boldsymbol{W}(x_3; \boldsymbol{k}, \omega)\, \mathrm{e}^{\mathrm{i}\psi}, \qquad \psi \equiv \boldsymbol{k} \cdot \boldsymbol{x} - \omega t, \tag{1}$$

with frequency ω, (surface) wave vector $\boldsymbol{k} = (k_1, k_2, 0)$ and phase speed $c \equiv \omega/k$, $(k \equiv |\boldsymbol{k}|)$ determined by a dispersion relation $D(\omega, \boldsymbol{k}) = 0$ (or, equivalently $\omega = \Omega(\boldsymbol{k})$). The function $\boldsymbol{W}(x_3; \boldsymbol{k}, \omega)$ describes the modal field, while the associated group velocity $\boldsymbol{c}_g \equiv \boldsymbol{g}(\boldsymbol{k})$ has components $g_i = \partial\Omega/\partial k_i = -(\partial D/\partial k_i)/(\partial D/\partial\omega)$, $(i = 1, 2)$, $g_3 = 0$ and has magnitude $c_g = |\boldsymbol{g}|$.

Strong nonlinear interaction between modes can arise if a second harmonic mode $\boldsymbol{E} = \boldsymbol{W}^{(2)}(x_3; \boldsymbol{k}^{(2)}, 2\omega) \exp(\mathrm{i}\psi^{(2)})$, where $\psi^{(2)} \equiv \boldsymbol{k}^{(2)} \cdot \boldsymbol{x} - 2\omega t$, has both $\boldsymbol{k}^{(2)} \simeq 2\boldsymbol{k}$ and $g_i^{(2)} \equiv \partial\Omega/\partial k_i^{(2)} \simeq g_i$ for $2\omega = \Omega(\boldsymbol{k}^{(2)})$. Thus, the phase and group velocities are simultaneously *closely matched*.

Coordinates $y_1 \equiv \varepsilon(p_1x_1 + p_2x_2 - c_gt)$, $y_2 \equiv \varepsilon(p_1x_2 - p_2x_1)$, $Z \equiv \varepsilon^2\,(p_1x_1 + p_2x_2)$ are introduced with $\boldsymbol{p} = (p_1, p_2, 0) \equiv \boldsymbol{g}/|\boldsymbol{g}|$ the unit vector parallel to the group velocity of the fundamental mode. Then, allowing for both spatial and temporal modulation of the complex amplitudes $A(y_1, y_2, Z)$, $B(y_1, y_2, Z)$ in the representation

$$\boldsymbol{E} = A\boldsymbol{W}\, \mathrm{e}^{\mathrm{i}\psi} + B\boldsymbol{W}^{(2)}\, \mathrm{e}^{2\mathrm{i}\psi} + \text{c.c.} + O(\varepsilon), \tag{2}$$

where c.c. denotes the complex conjugate, we find that the amplitude-modulation equations can be put into the form

$$\begin{aligned} &\mathrm{i}M\frac{\partial A}{\partial Z} + P_{JL}\frac{\partial^2 A}{\partial y_J \partial y_L} = J^* 2A^* B, \\ &\mathrm{i}\tilde{M}\frac{\partial B}{\partial Z} + \mathrm{i}\tilde{\gamma}_J\frac{\partial B}{\partial y_J} + \tilde{\Delta}B + \tilde{P}_{JL}\frac{\partial^2 B}{\partial y_J \partial y_L} = JA^2, \end{aligned} \tag{3}$$

where summation over repeated indices $J, L\ (= 1, 2)$ is assumed. Derivation of Equations (3) requires the restrictions $(\boldsymbol{g}^{(2)} - \boldsymbol{g})/|\boldsymbol{g}| = O(\varepsilon)$, $(\boldsymbol{k}^{(2)} - 2\boldsymbol{k})/k = O(\varepsilon^2)$, for some $\varepsilon \ll 1$. Thus, phase-velocity matching is more stringent than group-velocity matching.

Dependence on y_1 and y_2 records modulations along and transverse, respectively, to rays having orientation $\boldsymbol{g}$, relative to an observer moving at velocity $\boldsymbol{g}$. Dependence on Z measures the gradual evolution along each ray. The parameters P_{JL}/M and $\tilde{P}_{JL}/\tilde{M}$ are matrices of dispersion and diffraction coefficients for the fundamental and second-harmonic modes, respectively. The parameters $\tilde{\Delta}/\tilde{M}$ and $\tilde{\gamma}_J/\tilde{M}$ arise from the phase speed mismatch $c^{(2)} - c$ and group velocity mismatch $\boldsymbol{g}^{(2)} - \boldsymbol{g}$, respectively. The (complex) coefficient J/M is computed as a *modal overlap* integral of the two modes, while * denotes a complex conjugate, with M and $\tilde{M}$ $(\simeq 2M)$ both real.

Travelling-wave solutions to the *cascading system* (3) are sought in the form

$$\sqrt{2}A = J^{-1}\, \mathrm{e}^{\mathrm{i}\theta} u(\zeta), \qquad 2B = J^{-1}\, \mathrm{e}^{2\mathrm{i}\theta} v(\zeta), \tag{4}$$

$$\zeta \equiv \kappa_J y_J - M^{-1}Z, \qquad \theta \equiv \beta_J y_J - M^{-1}\sigma Z. \tag{5}$$

Insertion into (3) then yields the simultaneous ordinary differential equations (o.d.e.)

$$pu''(\zeta) + \mathrm{i}\Pi u'(\zeta) + \Sigma u(\zeta) = u^*(\zeta)v(\zeta), \tag{6}$$

$$\tilde{p}v''(\zeta) + \mathrm{i}\tilde{\Pi}v'(\zeta) + \tilde{\Sigma}v(\zeta) = u^2(\zeta) \tag{7}$$

in which $'$ denotes $\mathrm{d}/\mathrm{d}\zeta$ and the real coefficients are

$$p \equiv P_{JL}\kappa_J\kappa_L, \qquad \tilde{p} \equiv \tilde{P}_{JL}\kappa_J\kappa_L,$$

$$\Pi \equiv 1 + 2P_{JL}\beta_J\kappa_L, \qquad \tilde{\Pi} \equiv \tilde{M}M^{-1} + 4\tilde{P}_{JL}\beta_J\kappa_L + \tilde{\gamma}_J\kappa_J,$$

$$\Sigma \equiv -\sigma - P_{JL}\beta_J\beta_L, \qquad \tilde{\Sigma} \equiv \tilde{\Delta} - 2\tilde{\gamma}_J\beta_J - 2\sigma\tilde{M}M^{-1} - 4\tilde{P}_{JL}\beta_J\beta_L.$$

Motivated by the solutions found in [1–5] (and by the large mismatch behaviour $\tilde{\Sigma} \to \infty$ in which (7) gives $v \simeq \tilde{\Sigma}^{-1}u^2$, so that (6) yields the NLS equation), solutions were sought in [6] in the form

$$u = aS^2 + bST + cS + dT + e, \qquad v = \tilde{a}S^2 + \tilde{b}ST + \tilde{c}S + \tilde{d}T + \tilde{e}, \tag{8}$$

in which $a, b, \ldots, \tilde{e}$ are *complex* constants, while $S \equiv \mathrm{sech}(r\zeta)$, $T \equiv \tanh(r\zeta)$. Insertion into the system (6), (7) then yielded two sets of nine algebraic equations for $a, b, \ldots, \tilde{e}$. Although this algebraic system is overdetermined, it possesses six classes of solution, which were identified by an exhaustive search procedure. They yield solutions (8) as follows:

(I) $\quad u = 6r^2\sqrt{p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{sech}^2\, r\zeta, \qquad v = -6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\,\mathrm{sech}^2\, r\zeta \quad$ for $p\tilde{p} > 0$.

These solutions (published previously in [1, 2, 3]) have both $|u|$ and $|v|$ in the sech^2 form of a KdV soliton. They require that the *propagation conditions* $\Pi = \tilde{\Pi} = 0$, $\Sigma = -4r^2p$, $\tilde{\Sigma} = -4r^2\tilde{p}$ are fulfilled.

(II) $\quad u = \mathrm{i}6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{sech}\, r\zeta\,\tanh r\zeta, \qquad v = 6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\,\mathrm{sech}^2\, r\zeta \quad$ for $p\tilde{p} < 0$.

These solutions, in which u is odd, are given in [3] and [5]. They require the propagation conditions $\Pi = \tilde{\Pi} = 0$, $\Sigma = -r^2p$, $\tilde{\Sigma} = 2r^2\tilde{p}$.

(III) $\quad u = (2\Sigma\tilde{\Sigma})^{1/2}\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{sech}\, r\zeta, \qquad v = -2r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\,\mathrm{sech}^2\, r\zeta \quad$ for $\Sigma\tilde{\Sigma} > 0$.

These solutions, given in [4], require propagation conditions $\Pi = \tilde{\Pi} = \tilde{p} = 0$, $\Sigma = -r^2p$. This is the case $v = \tilde{\Sigma}^{-1}u^2$ in which (6) reduces to the o.d.e. for NLS solitons. Each of (I)–(III) describes bright-bright solitons.

(IV) $\quad u = 6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\{X + \mathrm{i}\tanh r\zeta\}\,\mathrm{sech}\, r\zeta,$

$\quad v = 6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\{\tfrac{1}{6}F_-(\mu) + \mathrm{sech}^2\, r\zeta + 2\,\mathrm{i}Y\tanh r\zeta\} \quad$ for $p\tilde{p} < 0$

with $X = \tan\mu + 3^{-1/2}\sec\mu$, $Y = \tan\mu$, $F_\pm(\mu) = X(9Y - 2X) \pm (2 + 3Y/X)$. In this family of solutions, given in [6], μ is adjustable. The propagation conditions are

$$\Pi/rp = 2(3Y - 2X), \qquad \tilde{\Pi}/r\tilde{p} = 2(3X - Y),$$

$$\Sigma/(r^2 p) = F_+(\mu) - 1, \qquad \tilde{\Sigma} = 0.$$

For each μ, the profile u is bright while $|v|$ tends to a nonzero constant as $|\zeta| \to \infty$.

$$\text{(V)} \quad u = e^{i\alpha}\sqrt{p\tilde{\Sigma}}\,\{\Pi/(2p) + ir \tanh r\zeta\}, \qquad v = \tilde{\Sigma}^{-1}u^2, \quad \text{for } p\tilde{\Sigma} > 0.$$

These solutions, mentioned in [6], like (III) correspond to a case in which (6) reduces to the o.d.e. for NLS solitons. They describe grey/grey solitons, with propagation conditions $\Sigma/(r^2 p) = 2 + \frac{1}{2}(\Pi/rp)^2$, $\tilde{p} = \tilde{\Pi} = 0$.

$$\text{(VI)} \quad u = 6r^2\sqrt{p\tilde{p}}\, e^{i\alpha}\{\text{sech}^2 r\zeta + iX \tanh r\zeta - QX\},$$

$$v = -6r^2 p\, e^{2i\alpha}\{\text{sech}^2 r\zeta + 2iY \tanh r\zeta + (Q^{-1} - Q)Y\} \qquad \text{for } p\tilde{p} > 0.$$

These solutions, in which typically both $|u|$ and $|v|$ are nonzero as $\zeta \to \pm\infty$, like (IV) involve an adjustable parameter, since X, Y and Q may be any solutions of

$$3QX - 6Q^{-1}Y + 9XY = 2 + 2X^2,$$

$$3QX^2 = 6XY(Q + Y) - 2Y^3 - 2Y - 3X^2 Y.$$

Typical profile pairs are shown in [6]. The associated propagation conditions are

$$\Pi/rp = 2(3Y - 2X), \qquad \tilde{\Pi}/r\tilde{p} = 2(3X - Y),$$

$$\Sigma/(r^2 p) = 6(QY + Q^{-1}X), \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 6QX^2Y^{-1}.$$

In solutions of each of the classes (I)–(VI) the propagation conditions provide restrictions on the parameters $\kappa_1, \kappa_2, \beta_1, \beta_2, \sigma$ and r appearing in (5), (8). Possibilities include spatial solitons ($\kappa_1 = 0$), in which $\varepsilon(\kappa_2 M)^{-1}$ corresponds to a 'walk-off' angle, purely temporal solitons ($\kappa_2 = 0$) and, more generally, temporal solitons oriented obliquely to the ray direction $\boldsymbol{p}$. In all cases, these are *vector solitons*, in which signals in the u and v modes are bound together by the cascading (feedback) process.

3. Periodic travelling-wave solutions

The procedure for seeking periodic solutions generalizing the solutions of classes (I)–(VI) to Equations (6) and (7) is to use the ansatz

$$\begin{aligned} u &= a + bN + cC + dD + eNC + fND + gCD + hN^2, \\ v &= \tilde{a} + \tilde{b}N + \tilde{c}C + \tilde{d}D + \tilde{e}NC + \tilde{f}ND + \tilde{g}CD + \tilde{h}N^2, \end{aligned} \tag{9}$$

involving the Jacobian elliptic functions

$$C \equiv \text{cn}\, r\zeta, \qquad D \equiv \text{dn}\, r\zeta, \qquad N \equiv \text{sn}\, r\zeta,$$

which satisfy the identities

$$C^2 + N^2 = 1, \qquad D^2 + \kappa^2 N^2 = 1,$$

$$C' = -rND, \qquad D' = -r\kappa^2 NC, \qquad N' = rCD, \tag{10}$$

where κ is the modulus and $'$ denotes $\mathrm{d}/\mathrm{d}\zeta$, as earlier. It should be recalled that C and D are even functions of $r\zeta$, while N is odd. Moreover, for moderate values of $|r\zeta|$, as $\kappa \to 1$ both $C, D \to \operatorname{sech} r\zeta$, while $N \to \tanh r\zeta$.

After the ansatz (9) is inserted into (6) and (7), use of the identities (10) to eliminate C^2 and D^2 in favour of N^2 allows both sides of each equation to be written as linear combinations of

$$1, N, C, D; \qquad NC, ND, CD, N^2; \qquad N^2C, N^2D, NCD, N^3;$$
$$N^3C, N^3D, N^2CD, N^4.$$

Comparison of coefficients in (6) then yields the sixteen equations:

$$\begin{aligned}
N^4 :&\quad 6\kappa^2 r^2 ph = -e^*\tilde{e} - \kappa^2 f^*\tilde{f} + \kappa^2 g^*\tilde{g} + h^*\tilde{h},\\
N^2CD :&\quad 6\kappa^2 r^2 pg = e^*\tilde{f} + f^*\tilde{e} + g^*\tilde{h} + h^*\tilde{g},\\
N^3D :&\quad 6\kappa^2 r^2 pf = f^*\tilde{h} + h^*\tilde{f} - e^*\tilde{g} - g^*\tilde{e},\\
N^3C :&\quad 6\kappa^2 r^2 pe = e^*\tilde{h} + h^*\tilde{e} - \kappa^2 f^*\tilde{g} - \kappa^2 g^*\tilde{f},\\
N^3 :&\quad 2\kappa^2 r^2 pb + \mathrm{i}2\kappa^2 r\Pi g = b^*\tilde{h} + h^*\tilde{b} - \kappa^2 d^*\tilde{f} - \kappa^2 f^*\tilde{d} - c^*\tilde{e} - e^*\tilde{c},\\
NCD :&\quad \mathrm{i}2r\Pi h = b^*\tilde{g} + g^*\tilde{b} + c^*\tilde{f} + f^*\tilde{c} + d^*\tilde{e} + e^*\tilde{d},\\
N^2D :&\quad 2\kappa^2 r^2 pd - \mathrm{i}2r\Pi e = b^*\tilde{f} + f^*\tilde{b} - c^*\tilde{g} - g^*\tilde{c} + d^*\tilde{h} + h^*\tilde{d},\\
N^2C :&\quad 2\kappa^2 r^2 pc - \mathrm{i}2\kappa^2 r\Pi f = b^*\tilde{e} + e^*\tilde{b} + c^*\tilde{h} + h^*\tilde{c} - \kappa^2 d^*\tilde{g} - \kappa^2 g^*\tilde{d},\\
N^2 :&\quad [\Sigma - 4(1+\kappa^2)r^2 p]\,h = a^*\tilde{h} + h^*\tilde{a} + b^*\tilde{b} - c^*\tilde{c} - \kappa^2 d^*\tilde{d}\\
&\qquad\qquad + e^*\tilde{e} + f^*\tilde{f} - (1+\kappa^2)g^*\tilde{g},\\
CD :&\quad [\Sigma - (1+\kappa^2)r^2 p]\,g + \mathrm{i}r\Pi b = a^*\tilde{g} + g^*\tilde{a} + c^*\tilde{d} + d^*\tilde{c},\\
ND :&\quad [\Sigma - (1+4\kappa^2)r^2 p]\,f - \mathrm{i}r\Pi c = a^*\tilde{f} + f^*\tilde{a} + b^*\tilde{d} + d^*\tilde{b} + e^*\tilde{g} + g^*\tilde{e},\\
NC :&\quad [\Sigma - (4+\kappa^2)r^2 p]\,e - \mathrm{i}\kappa^2 r\Pi d = a^*\tilde{e} + e^*\tilde{a} + b^*\tilde{c} + c^*\tilde{b} + f^*\tilde{g} + g^*\tilde{f},\\
D :&\quad (\Sigma - \kappa^2 r^2 p)d + \mathrm{i}r\Pi e = a^*\tilde{d} + d^*\tilde{a} + c^*\tilde{g} + g^*\tilde{c},\\
C :&\quad (\Sigma - r^2 p)c + \mathrm{i}r\Pi f = a^*\tilde{c} + c^*\tilde{a} + d^*\tilde{g} + g^*\tilde{d},\\
N :&\quad [\Sigma - (1+\kappa^2)r^2 p]\,b - \mathrm{i}(1+\kappa^2)r\Pi g = a^*\tilde{b} + b^*\tilde{a} + c^*\tilde{e} + e^*\tilde{c} + d^*\tilde{f} + f^*\tilde{d},\\
1 :&\quad \Sigma a + 2r^2 ph = a^*\tilde{a} + c^*\tilde{c} + d^*\tilde{d} + g^*\tilde{g}.
\end{aligned}$$

From Equation (7) an analogous set of 16 equations is obtained, with $a, b, \dots h;\ p, \Pi, \Sigma$ on the left-hand sides replaced by $\tilde{a}, \tilde{b}, \dots \tilde{h};\ \tilde{p}, \tilde{\Pi}, \tilde{\Sigma}$ and with all symbols * and $\tilde{}$ removed from the right-hand sides. This (grossly) overdetermined set of 32 equations for sixteen coefficients in (9) and for p, $\tilde{p}$, Π, $\tilde{\Pi}$, Σ and $\tilde{\Sigma}$ may be analysed (much as the analogous equations arising from (9) are analysed in [6]) by suitable groupings of the equations and coefficients. Moreover, it is known from the solutions presented in Section 2, that many classes of vector soliton exist in the limit $\kappa = 1$.

The eight equations arising from the coefficients of N^4, N^2CD, N^3D and N^3C may be combined in pairs to give

$$
\begin{aligned}
6\kappa^2 r^2 \tilde{p}(\tilde{e} \pm \kappa \tilde{f}) &= 2(e \pm \kappa f)(h \mp \kappa g),\\
6\kappa^2 r^2 \tilde{p}(\tilde{h} \pm \kappa \tilde{g}) &= (h \pm \kappa g)^2 - (e \mp \kappa f)^2,\\
6\kappa^2 r^2 p(h \pm \kappa g) &= (h^* \pm \kappa g^*)(\tilde{h} \pm \kappa \tilde{g}) - (e^* \mp \kappa f^*)(\tilde{e} \mp \kappa \tilde{f}),\\
6\kappa^2 r^2 p(e \pm \kappa f) &= (e^* \pm \kappa f^*)(\tilde{h} \mp \kappa \tilde{g}) + (h^* \mp \kappa g^*)(\tilde{e} \pm \kappa \tilde{f}).
\end{aligned}
$$

Moreover, these may be further combined and rewritten in terms of the eight quantities

$$
\begin{aligned}
h + \kappa g \pm \mathrm{i}(e - \kappa f) &\equiv X_\pm, \qquad h - \kappa g \pm \mathrm{i}(e + \kappa f) \equiv Y_\pm,\\
\tilde{h} + \kappa \tilde{g} \pm \mathrm{i}(\tilde{e} - \kappa \tilde{f}) &\equiv \tilde{X}_\pm, \qquad \tilde{h} - \kappa \tilde{g} \pm \mathrm{i}(\tilde{e} + \kappa \tilde{f}) \equiv \tilde{Y}_\pm,
\end{aligned}
$$

as two uncoupled, but similar, pairs of equations

$$
\begin{aligned}
&6\kappa^2 r^2 \tilde{p} \tilde{X}_\pm = X_\pm^2, \qquad 6\kappa^2 r^2 p X_\pm = X_\pm^* \tilde{X}_\pm,\\
&6\kappa^2 r^2 \tilde{p} \tilde{Y}_\pm = Y_\pm^2, \qquad 6\kappa^2 r^2 p Y_\pm = Y_\pm^* \tilde{Y}_\pm.
\end{aligned}
$$

These show, that either $X_+ = 0 = X_-$ or $X_+ X_-^* = K^2 = X_- X_+^*$, while simultaneously either $Y_+ = 0 = Y_-$ or $Y_+ Y_-^* = K^2 = Y_- Y_+^*$, where $K \equiv 6\kappa^2 r^2 \sqrt{p\tilde{p}}$ is either real or pure imaginary. Consequently, four possibilities arise for $e, f, g, h; \tilde{e}, \tilde{f}, \tilde{g}, \tilde{h}$:

(i) $X_\pm = 0 = Y_\pm$. This gives $e = f = g = h = 0$ with either $\tilde{e} = \tilde{f} = \tilde{g} = \tilde{h}$ or $\tilde{p} = 0$ (a propagation condition).

(ii) $X_\pm = 0, Y_\pm = K\rho^{\pm 1}\,\mathrm{e}^{\mathrm{i}\alpha}$ with ρ and α real and arbitrary, so that $\tilde{X}_\pm = 0, \tilde{Y}_\pm = 6\kappa^2 r^2 p \rho^{\pm 2}\,\mathrm{e}^{2\mathrm{i}\alpha}$. This yields the representation

$$
\begin{aligned}
e = \kappa f &= -\tfrac{3}{2}\mathrm{i}\kappa^2 r^2 \sqrt{p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}(\rho - \rho^{-1}), \qquad h = -\kappa g = \tfrac{3}{2}\kappa^2 r^2 \sqrt{p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}(\rho + \rho^{-1}),\\
\tilde{e} = \kappa \tilde{f} &= -\tfrac{3}{2}\mathrm{i}\kappa^2 r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}(\rho^2 - \rho^{-2}), \qquad \tilde{h} = -\kappa \tilde{g} = \tfrac{3}{2}\kappa^2 r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}(\rho^2 + \rho^{-2}).
\end{aligned}
$$

(iii) $Y_\pm = 0$ with $X_\pm = K\rho^{\pm 1}\,\mathrm{e}^{\mathrm{i}\alpha}$. Since this is equivalent to (ii) after the replacement $\kappa \to -\kappa$, this need not be considered further.

(iv) $X_\pm = K{\rho_1}^{\pm 1}\mathrm{e}^{\mathrm{i}\alpha}, \qquad \tilde{X}_\pm = 6\kappa^2 r^2 p {\rho_1}^{\pm 2}\,\mathrm{e}^{2\mathrm{i}\alpha}$;
$Y_\pm = K{\rho_2}^{\pm 1}\,\mathrm{e}^{\mathrm{i}\beta}, \qquad \tilde{Y}_\pm = 6\kappa^2 r^2 p {\rho_2}^{\pm 2}\,\mathrm{e}^{2\mathrm{i}\beta}$

with ρ_1, ρ_2, α and β real and arbitrary. This gives equations for each of $e, f, \ldots \tilde{h}$ which involve all four parameters α, β, ρ_1 and ρ_2 (*e.g.* $h = \frac{1}{4}(X_+ + X_- + Y_+ + Y_-)$).

An exhaustive search procedure next solves the eight equations arising from terms of degree three in C, D and N as a linear system for $b, c, d, \tilde{b}, \tilde{c}, \tilde{d}$, together with compatibility conditions. However, since in the limit $\kappa \to 1$ solutions have been found only for classes (I)–(VI) in each of which many of the coefficients in (9) vanish, a simpler strategy has been followed, by seeking solutions in classes which generalize each of (I)–(VI).

Class I
Since the only combinations of C, D and N which appear in (9) yet tend to $\mathrm{sech}^2\, r\zeta$ as $\kappa \to 1$ are CD and $1 - N^2$, solutions are sought by retaining only the coefficients a, g, h, $\tilde{a}$, $\tilde{g}$ and $\tilde{h}$ in (9). This yields sixteen algebraic equations having three separate possibilities for solution (each with $p\tilde{p} > 0$):

I.1. This utilizes (ii) or (iii) with $\rho = 1$, with α arbitrary and with $h = \pm\kappa g$, $\tilde{h} = \pm\kappa\tilde{g}$. The profiles are:

$$u = 3r^2\sqrt{p\tilde{p}}\, \mathrm{e}^{\mathrm{i}\alpha}\{\kappa^2\mathrm{sn}^2\, r\zeta \pm \kappa\, \mathrm{cn}\, r\zeta\, \mathrm{dn}\, r\zeta - \tfrac{1}{6}[1 + \kappa^2 + L]\},$$

$$v = 3r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}\{\kappa^2\mathrm{sn}^2\, r\zeta \pm \kappa\, \mathrm{cn}\, r\zeta\, \mathrm{dn}\, r\zeta - \tfrac{1}{6}[1 + \kappa^2 + L]\},$$

where $L \equiv \pm\sqrt{(1 + \kappa^2)^2 + 12\kappa^2}$. The corresponding *propagation conditions* are:

$$\Pi = \tilde{\Pi} = 0, \qquad \Sigma/(r^2 p) = \tilde{\Sigma}/(r^2\tilde{p}) = -L.$$

I.2. This has $g = \tilde{g} = 0$, $h = 6\kappa^2 r^2\sqrt{p\tilde{p}}\, \mathrm{e}^{\mathrm{i}\alpha}$, $\tilde{h} = 6\kappa^2 r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}$ with α arbitrary. The profiles are:

$$u = 6r^2\sqrt{p\tilde{p}}\, \mathrm{e}^{\mathrm{i}\alpha}\{\kappa^2\, \mathrm{sn}^2\, r\zeta - \tfrac{1}{3}[1 + \kappa^2 + M]\},$$
$$v = 6r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}\{\kappa^2\, \mathrm{sn}^2\, r\zeta - \tfrac{1}{3}[1 + \kappa^2 + M]\},$$

where $M = \pm\sqrt{1 - \kappa^2 + \kappa^4}$ and the *propagation conditions* are:

$$\Pi = \tilde{\Pi} = 0, \qquad \Sigma/(r^2 p) = \tilde{\Sigma}/(r^2\tilde{p}) = -4M.$$

I.3. This has $h = \tilde{g} = 0$, $g = \mathrm{i}6\kappa^2 r^2\sqrt{p\tilde{p}}\, \mathrm{e}^{\mathrm{i}\alpha}$, $\tilde{h} = -6\kappa^2 r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}$ with α arbitrary. The profiles are:

$$u = \mathrm{i}6\kappa r^2\sqrt{p\tilde{p}}\, \mathrm{e}^{\mathrm{i}\alpha}\mathrm{cn}\, r\zeta\, \mathrm{dn}\, r\zeta, \qquad v = 6\kappa^2 r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}[2(1 + \kappa^2)^{-1} - \mathrm{sn}^2\, r\zeta],$$

with $\Pi = \tilde{\Pi} = 0$, $\Sigma/(r^2 p) = 1 + \kappa^2 - 12\kappa^2(1 + \kappa^2)^{-1}$, $\tilde{\Sigma}/(r^2\tilde{p}) = -2(1 + \kappa^2)$.

To help in the interpretation of these profiles, it should be recalled that each of $\mathrm{cn}\, r\zeta$, $\mathrm{dn}\, r\zeta$, $\mathrm{sn}\, r\zeta$ has period $4K(\kappa)$, where $K(\kappa)$ is the complete elliptic integral. Moreover, cn and dn are even functions, while sn is odd and $\mathrm{dn}(r\zeta + 2K) = \mathrm{dn}\, r\zeta$, while $\mathrm{cn}(r\zeta + 2K) = -\mathrm{cn}\, r\zeta$ and $\mathrm{sn}(r\zeta + 2K) = -\mathrm{sn}\, r\zeta$. Hence, changing the sign of any term $\kappa\, \mathrm{cn}\, r\zeta$ in I.1 or I.3 is equivalent merely to a shift $2K$ along the $r\zeta$ axis.

For I.1, the profiles $u\, \mathrm{e}^{-\mathrm{i}\alpha}$ and $v\, \mathrm{e}^{-2\mathrm{i}\alpha}$ are real and in constant ratio. For L positive, in the limit $\kappa \to 1$, both $-u\, \mathrm{e}^{-\mathrm{i}\alpha}$ and $-v\, \mathrm{e}^{-2\mathrm{i}\alpha}$ are $\mathrm{sech}^2\, r\zeta$ pulses when the minus sign is taken in front of κ. For $\kappa \neq 1$, the profiles become periodically spaced pulses, with period $4K(\kappa)$. When the positive sign is taken in front of κ, the only change is that pulses are displaced by $2K(\kappa)$ along the $r\zeta$ axis (so that, as $\kappa \to 1$, then $u \to 0$, $v \to 0$ at $\zeta = 0$). These pulses I.1_+ are illustrated in Figure 1(a).

The choice of L as the negative square root merely adds $r^2\sqrt{p\tilde{p}}\, |L|$ to $u\, \mathrm{e}^{-\mathrm{i}\alpha}$, so giving the profiles I.1_- as shown in Figure 1(b). Again $v\, \mathrm{e}^{-2\mathrm{i}\alpha}$ has a similar profile, so that the solution describes periodically-spaced twin-hole dark pulses. These generalize the twin-hole

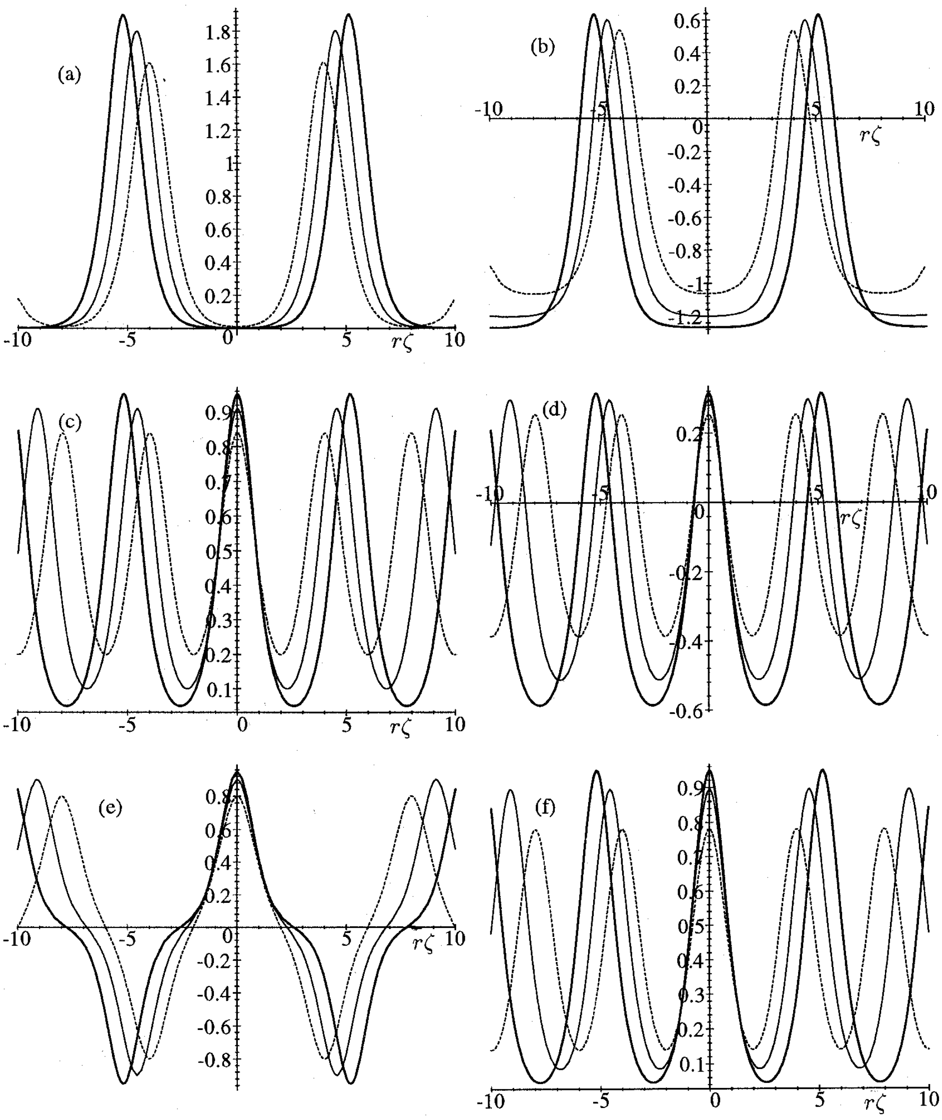

Figure 1. Profiles of $\bar{u} \equiv u/(6r^2\sqrt{p\bar{p}}\,\mathrm{e}^{\mathrm{i}\alpha})$ and $\bar{v} \equiv v/(6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha})$ versus $r\zeta$ for $\kappa = 0{\cdot}8$ ----, $\kappa = 0{\cdot}9$ —, $\kappa = 0{\cdot}99$ —. (a) Case I.1$_+$, $-\bar{u} = -\bar{v}$; (b) Case I.1$_-$, $-\bar{u} = -\bar{v}$; (c) Case I.2$_-$, $-\bar{u} = -\bar{v}$; (d) Case I.2$_-$, $-\bar{u} = -\bar{v}$; (e) Case I.3, $-\mathrm{i}\bar{u}$; (f) Case I.3, $\bar{v}$.

dark pulses of Hayata and Koshiba [10] and Buryak and Kivshar [11]. They have no change in phase across each pulse.

For I.2, the profiles of $u\,\mathrm{e}^{-\mathrm{i}\alpha}$ and $v\,\mathrm{e}^{-2\mathrm{i}\alpha}$ are again real and in constant ratio. Taking M as the positive square root gives profiles I.2$_+$ of $-u\,\mathrm{e}^{-\mathrm{i}\alpha}$ and $-v\,\mathrm{e}^{-2\mathrm{i}\alpha}$ as periodically spaced pulses, but with period $2K(\kappa)$ which is half that of I.1. Similarly, the choice of M as the negative square root gives periodically-spaced twin-hole pulses, without phase change. These profiles (I.2$_-$) are shown in Figures 1(c), (d), respectively.

For I.3, profiles of $-\mathrm{i}u\,\mathrm{e}^{-\mathrm{i}\alpha}$ and $v\,\mathrm{e}^{-2\mathrm{i}\alpha}$ are both real and periodic, but they differ. The profile of $v\,\mathrm{e}^{-2\mathrm{i}\alpha}$ is a sequence of bright pulses with period $2K(\kappa)$, while that of $-\mathrm{i}u\,\mathrm{e}^{-\mathrm{i}\alpha}$ has double the period. The alternation of positive and negative pulses corresponds to phase

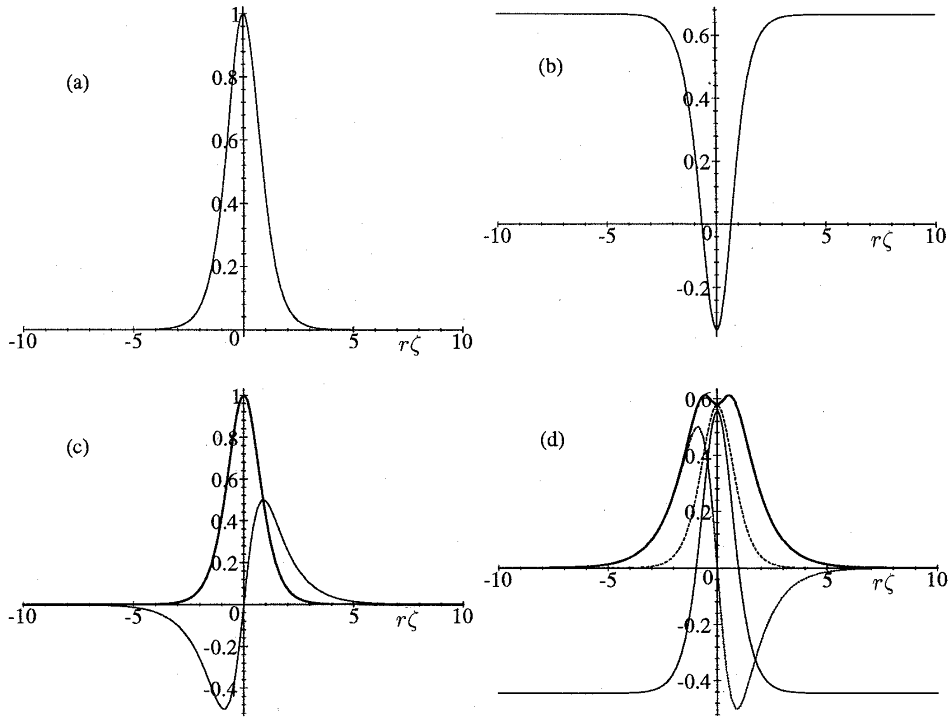

Figure 2. Solitary wave profiles. (a) $\bar{u} = -\bar{v} = \mathrm{sech}^2 r\zeta$ for Class (I); (b) $\bar{u} = \bar{v} = \frac{2}{3} - \mathrm{sech}^2 r\zeta$ twin-hole dark solitons [10, 11]; (c) $-\mathrm{i}\bar{u} = \mathrm{sech}\, r\zeta \tanh r\zeta$, $\bar{v} = \mathrm{sech}^2 r\zeta$ for Class (II); (d) $\mathfrak{Re}(\bar{u})$ - - - -, $\mathfrak{Im}(\bar{u}) \cdots\cdots$, $|\bar{u}|$ — and $\bar{v}$ —, for the special case $Y = 0$ of Class (IV) with $\bar{u} = (3^{-1/2} - \mathrm{i} \tanh r\zeta)\mathrm{sech}\, r\zeta$, $\bar{v} = \mathrm{sech}^2 r\zeta - \frac{4}{9}$.

shifts of $\pm\pi$ across u pulses at separation $2K(\kappa)$ (without a phase shift in $v\,\mathrm{e}^{-2\mathrm{i}\alpha}$). Thus, although the intensities $|u|^2$ and $|v|^2$ are somewhat similar to those for I.2$_+$, there is an essential distinction in the phase dependence.

For each of I.1, I.2 and I.3 the propagation conditions include the equations $\Pi = \tilde{\Pi} = 0$, while $\Sigma/(r^2 p)$ and $\tilde{\Sigma}/(r^2\tilde{p})$ depend on κ. Since in all the cases I.1$_+$, I.2$_+$ and I.3 corresponding to periodically spaced bright pulses it is seen that $\Sigma/(r^2 p) \to -4$, $\tilde{\Sigma}/(r^2\tilde{p}) \to -4$ as $\kappa \to 1$, each of these cases is a generalization of the Karamzin and Sukhorukov [1] solution (I). As $\kappa \to 1$, the period increases (logarithmically) and the profiles tend to periodically spaced copies of solution (I), shown in Figure 2(a). The twin-hole cases I.1$_-$ and I.2$_-$ have $L \to -4$, $M \to -1$, so that $\Sigma/(r^2 p) \to 4$, $\tilde{\Sigma}/(r^2\tilde{p}) \to 4$ as $\kappa \to 1$. Moreover, since they have

$$-\mathrm{i}u\,\mathrm{e}^{-\mathrm{i}\alpha}, \quad v\,\mathrm{e}^{-2\mathrm{i}\alpha} \to \tanh^2 r\zeta - \tfrac{1}{3},$$

they are natural generalizations of the Hayata and Koshiba [10] twin-hole dark solitons shown in Figure 2(b), included in (VI) as the limiting case $Y = X$, $QX = 2/3$, $X \to 0$.

Class II

To generalize the Werner and Drummond odd–even solutions (II) shown in Figure 2(c), the ansatz (9) is simplified by retaining only the coefficients e, f, $\tilde{a}$, $\tilde{g}$ and $\tilde{h}$. This again produces sixteen algebraic equations which possess three types of solution. Since they each correspond to $p\tilde{p} < 0$, the representation in (iv) is modified using

$$X_\pm = \pm 6\kappa^2 r^2 \sqrt{-p\tilde{p}}\, \rho_1{}^{\pm 1}\, \mathrm{e}^{\mathrm{i}\alpha}, \qquad Y_\pm = \pm 6\kappa^2 r^2 \sqrt{-p\tilde{p}}\, \rho_2{}^{\pm 1}\, \mathrm{e}^{\mathrm{i}\beta}.$$

II.1. This has $f = \tilde{g} = 0$, so using $\rho_1 = \rho_2 = -1$, $\beta = \alpha$. The profiles are

$$u = \mathrm{i}6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\kappa^2 \,\mathrm{sn}\, r\zeta \;\mathrm{cn}\, r\zeta, \qquad v = 6r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}\kappa^2\{(2-\kappa^2)^{-1} - \mathrm{sn}^2\, r\zeta\}$$

with corresponding propagation conditions

$$\Pi = \tilde{\Pi} = 0; \qquad \Sigma/(r^2 p) = 4 + \kappa^2 - 6\kappa^2(2-\kappa^2)^{-1}, \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 4 - 2\kappa^2.$$

II.2. This has $e = \tilde{g} = 0$ and arises from $\rho_1 = -\rho_2 = 1$, $\beta = \alpha$. The profiles are:

$$u = \mathrm{i}6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\kappa \,\mathrm{sn}\, r\zeta \;\mathrm{dn}\, r\zeta, \qquad v = 6r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}\kappa^2\{(2\kappa^2 - 1)^{-1} - \mathrm{sn}^2\, r\zeta\}$$

with corresponding propagation conditions

$$\Pi = \tilde{\Pi} = 0; \quad \Sigma/(r^2 p) = 1 + 4\kappa^2 - 6\kappa^2(2\kappa^2-1)^{-1}, \quad \tilde{\Sigma}/(r^2\tilde{p}) = 4\kappa^2 - 2.$$

II.3. This has $e \mp \kappa f = 0, \tilde{h} \pm \kappa\tilde{g} = 0$ and arises from (ii) or (iii) respectively, after modification to allow for $p\tilde{p}$ being negative. The profiles are:

$$u = \mathrm{i}3r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\kappa \,\mathrm{sn}\, r\zeta\; \{\kappa\,\mathrm{cn}\, r\zeta \pm \mathrm{dn}\, r\zeta\},$$

$$v = 3r^2 p\, \mathrm{e}^{2\mathrm{i}\alpha}\{2\kappa^2(1+\kappa^2)^{-1} \pm \kappa\,\mathrm{cn}\, r\zeta\,\mathrm{dn}\, r\zeta - \kappa^2 \mathrm{sn}\, r\zeta\}$$

with corresponding propagation conditions

$$\Pi = \tilde{\Pi} = 0; \qquad \Sigma/(r^2 p) = 1 + \kappa^2 - 6\kappa^2(1+\kappa^2)^{-1}, \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 1 + \kappa^2.$$

In each case, u is an odd, imaginary function, while v is even and real. In II.1 both u and v have fundamental period $2K(\kappa)$; in II.2, v still has period $2K(\kappa)$, but u has fundamental period $4K(\kappa)$; in II.3, both u and v have fundamental period $4K(\kappa)$. As we see in Figure 3(e), (f), case II.3$_+$ (using the plus sign in $\pm$) describes widely-spaced odd–even pulses resembling the vector solitons of Class (II) (see Figure 2(c)) and centred at $r\zeta = 0, \pm 4K(\kappa)$, etc. The analogous case II.3$_-$ differs only by displacing the pulses to $\pm 2K(\kappa)$, etc. Case II.1 has similar pulses but at closer spacing $2K(\kappa)$. It is noticeable in Figure 3(b) that, even for $\kappa = 0{\cdot}9$, this $|v|$ profile develops appreciable brightness within the dark troughs between pulses. Case II.2 differs in that the odd pulses of u which are centred at $r\zeta = (4n+2)K(\kappa)$ have opposite sense to those at $r\zeta = 4nK(\kappa)$, n integer, as seen in Figure 3(c). Correspondingly, the pulses for v in Figure 3(d) do not develop twin-holes as in case II.1. Unlike cases II.1 and II.3, in which the trajectories of $(-\mathrm{i}u\,\mathrm{e}^{-\mathrm{i}\alpha}, v\,\mathrm{e}^{-2\mathrm{i}\alpha})$ encircle the origin once per period, case II.2 has trajectories which are open arcs. Consequently, there is no relative shift in phase per complete period of the profile.

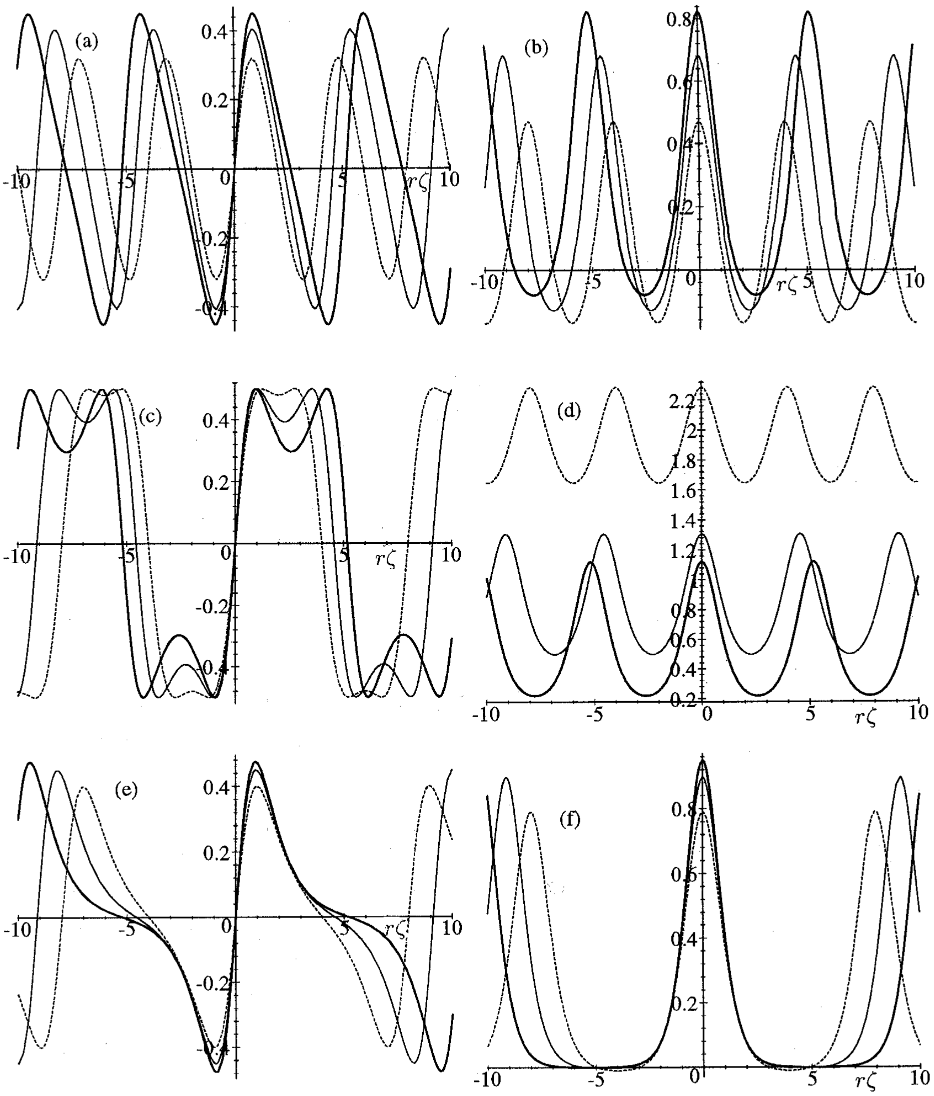

Figure 3. Profiles of $\bar{u} = -\mathrm{i}u/(6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha})$ and $\bar{v} = v/(6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha})$ versus $r\zeta$ for $\kappa = 0{\cdot}8$ - - - -, $\kappa = 0.9$ —, $\kappa = 0{\cdot}99$ —. (a) Case II.1, $\bar{u}$; (b) Case II.1, $\bar{v}$; (c) Case II.2, $\bar{u}$; (d) Case II.1, $\bar{v}$; (e) Case II.3$_+$, $\bar{u}$; (f) Case II.3$_+$, $\bar{v}$.

Class III

This is the case in which $v = \tilde{\Sigma}^{-1}u^2$, with c, d, $\tilde{a} = \tilde{\Sigma}^{-1}(c^2 + d^2)$, $\tilde{g} = 2\tilde{\Sigma}^{-1}cd$ and $\tilde{h} = -\tilde{\Sigma}^{-1}(c^2 + \kappa^2 d^2)$ the only nonvanishing coefficients. These solutions have profiles

$$u = \mathrm{i}r\sqrt{\frac{-p\tilde{\Sigma}}{2}}\,\mathrm{e}^{\mathrm{i}\alpha}\{\kappa\,\mathrm{cn}\,r\zeta \pm \mathrm{dn}\,r\zeta\},$$

$$v = r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\{\tfrac{1}{2}(1+\kappa^2) \pm \kappa\,\mathrm{cn}\,r\zeta\,\mathrm{dn}\,r\zeta - \kappa^2\,\mathrm{sn}^2\,r\zeta\}$$

with corresponding propagation conditions

$$\Pi = \tilde{\Pi} = 0; \qquad \tilde{p} = 0, \qquad \Sigma/(r^2 p) = -\tfrac{1}{2}(1+\kappa^2).$$

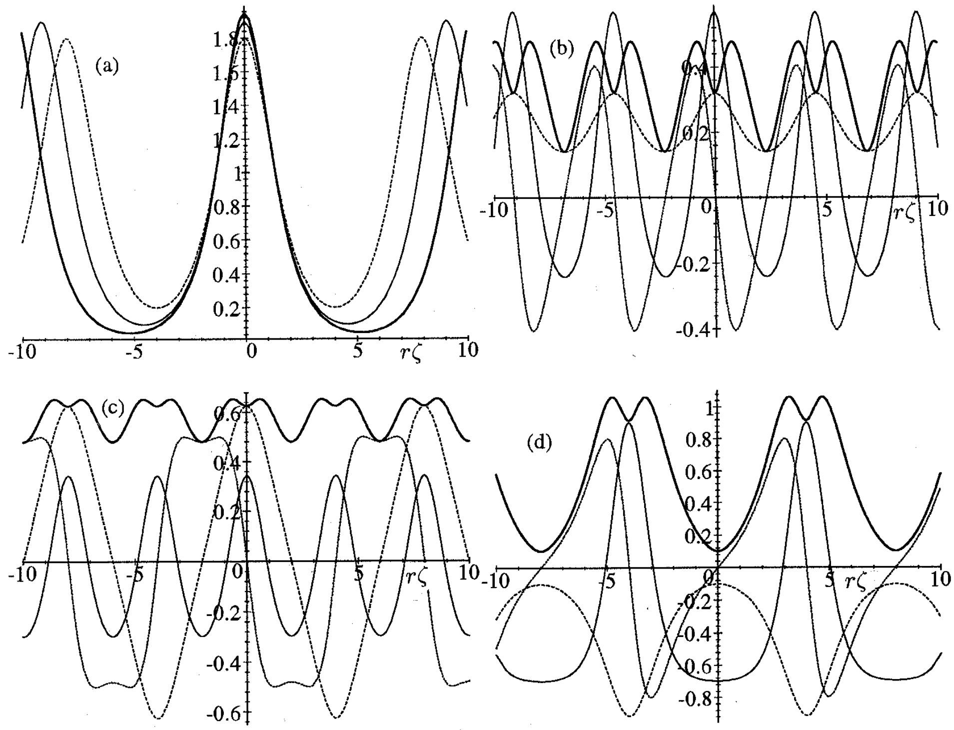

Figure 4. (a) Profiles $\kappa\,\mathrm{cn}\,r\zeta + \mathrm{dn}\,r\zeta$ for Class III, $\kappa = 0{\cdot}8$ - - - -, $\kappa = 0{\cdot}9$ —, $\kappa = 0{\cdot}99$ —. (b–d) Profiles of $\mathfrak{Re}(\bar{u})$ - - - -, $\mathfrak{Im}(\bar{u})$ · · · · · ·, $|\bar{u}|$ — and $\bar{v}$ — for Class IV with $\bar{u} \equiv u/(6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha})$, $\bar{v} \equiv v/(6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha})$ versus $r\zeta$ for $\kappa = 0{\cdot}9$. (b) Case IV.1; (c) Case IV.2; (d) Case IV.3.

The profiles are even and represent bright pulses at separation $4K(\kappa)$ (centred at $4nK(\kappa)$ for the positive sign as in Figure 4(a), but centred at $(2 + 4n)K(\kappa)$ for the negative sign. These latter have profile for v virtually identical to those for Case I.1_+). However, like the corresponding solitary pulses (III), found by Werner and Drummond [4], these solutions correspond to the abnormal case in which dispersion and diffraction of the second-harmonic are effectively absent. Their practical importance is thereby limited.

Class IV

Solutions generalizing the bright-dark solitary waves (IV) require that, in (9), only the coefficients $c, d, e, f, \tilde{a}, \tilde{b}, \tilde{g}$ and $\tilde{h}$ are retained. Besides the bright-bright possibilities II.1–II.3 found earlier, this yields three cases (each for $p\tilde{p} < 0$):

IV.1. This has $c = f = \tilde{b} = \tilde{g} = 0$, so using $\rho_1 = \rho_2 = 1$, we have $\beta = \alpha$. The profiles are:

$$u = 6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\{H_1\mathrm{dn}\,r\zeta - \mathrm{i}\kappa^2\mathrm{cn}\,r\zeta\;\mathrm{sn}\,r\zeta\},$$

$$v = 6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\{\tfrac{1}{3}(1 + \kappa^2 - H_1^2) - \kappa^2\mathrm{sn}^2 r\zeta\}$$

where $H_1^2 = \frac{1}{3}\{\kappa^2 - 2 + \sqrt{(2\kappa^2 + 2)(2\kappa^2 - 1)}\}$, for $\kappa^2 \in [\sqrt{3} - 1, 1]$. The corresponding propagation conditions are $\Pi/(rp) = 4H_1$, $\tilde{\Pi}/(r\tilde{p}) = -6H_1$,

$$\Sigma/(r^2 p) = 2 - \kappa^2 - 2H_1^2, \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 4 - 2\kappa^2 - 6H_1^2.$$

IV.2. This case has instead $d = e = \tilde{b} = \tilde{g} = 0$ and $\rho_1 = -\rho_2 = -1$, $\beta = \alpha$. The profiles are:

$$u = 6r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}\kappa\,\{H_2\,\mathrm{cn}\,r\zeta - \mathrm{i}\,\mathrm{sn}\,r\zeta\,\mathrm{dn}\,r\zeta\},$$

$$v = 6r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\{\tfrac{1}{3}(1+\kappa^2 - H_2^2) - \kappa^2\,\mathrm{sn}^2\,r\zeta\},$$

where $H_2^2 = \frac{1}{3}\{1 - 2\kappa^2 + \sqrt{(2\kappa^2+2)(2-\kappa^2)}\}$, for any $\kappa^2 \leqslant 1$. The corresponding propagation conditions are $\Pi/(rp) = 4H_2$, $\tilde{\Pi}/(r\tilde{p}) = -6H_2$,

$$\Sigma/(r^2p) = 2\kappa^2 - 1 - 2H_2^2, \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 4\kappa^2 - 2 - 6H_2^2.$$

IV.3. This has $e \mp \kappa f = \tilde{h} \pm \kappa\tilde{g} = 0$ and arises from (ii) or (iii) respectively. The profiles are:

$$u = 3r^2\sqrt{-p\tilde{p}}\,\mathrm{e}^{\mathrm{i}\alpha}(\kappa\,\mathrm{cn}\,r\zeta \mp \mathrm{dn}\,r\zeta)(\tfrac{1}{2}H_3 - \mathrm{i}\kappa\,\mathrm{sn}\,r\zeta),$$

$$v = 3r^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\{\tfrac{1}{6}(1+\kappa^2 - H_3^2) \mp \kappa\,\mathrm{cn}\,r\zeta\,\mathrm{dn}\,r\zeta - \kappa^2\mathrm{sn}^2 r\zeta\},$$

where $H_3^2 = \frac{1}{3}\{-2 - 2\kappa^2 + \sqrt{2(6\kappa+\kappa^2+1)(6\kappa-\kappa^2-1)}\}$, for $\kappa^2 \in [5-2\sqrt{6}, 1]$ and the propagation conditions are $\Pi/(rp) = 2H_3$, $\tilde{\Pi}/(r\tilde{p}) = -3H_3$,

$$\Sigma/(r^2p) = \tfrac{1}{2}(1+\kappa^2 - H_3^2), \qquad \tilde{\Sigma}/(r^2\tilde{p}) = 1 + \kappa^2 - \tfrac{3}{2}H_3^2.$$

For all cases, $u\,\mathrm{e}^{-\mathrm{i}\alpha}$ is complex, $\mathfrak{Re}(u\,\mathrm{e}^{-\mathrm{i}\alpha})$ and $\mathfrak{Im}(u\,\mathrm{e}^{-\mathrm{i}\alpha})$ are even and odd functions, respectively, while $v\,\mathrm{e}^{-2\mathrm{i}\alpha}$ is even and real. They are shown in Figures 4(b), (c), (d) and have periods for u and v which are the same as for the corresponding cases of Class II. The intensities of these waves look similar and represent a sequence of double bright pulses for $|u|$ and of bright-centred twin-hole pulses for $|v|$. As $\kappa \to 1$, each of these profiles reduces to the single case of the solitary wave (IV) which has zero phase change, namely the case $\mu = 0$ $(Y = 0)$ of Figure 2(d).

We should note, that Cases II.1 and II.2, like Cases IV.1 and IV.2 are connected by a reciprocal transformation. Namely, by applying the transformation $(r,\kappa) \to (\kappa^{-1}r, \kappa^{-1})$ to one member of each pair, one can obtain the formulae for the other member.

The only solution generalizing Class (V) relates to the periodic solution of the NLS equation. It is

$$u = r\kappa\sqrt{2p\tilde{\Sigma}}\,\mathrm{e}^{\mathrm{i}\alpha}\,\mathrm{sn}\,r\zeta, \qquad v = 2r^2\kappa^2 p\,\mathrm{e}^{2\mathrm{i}\alpha}\,\mathrm{sn}^2\,r\zeta,$$

with $\Pi = \tilde{\Pi} = \tilde{p} = 0$, $\Sigma = r^2(1+\kappa^2)p$. Also, despite extensive use of Maple V, we have found no generalizations of Class (VI) other than the twin-hole solutions found earlier as cases $\mathrm{I.1}_-$, $\mathrm{I.2}_-$, $\mathrm{I.3}_-$.

We conclude by observing that, while we have been able to generalize vector solitons of each of the Classes (I–III) to periodic solutions, the only extensions of Class (IV) found are generalizations of the special case illustrated in Figure 2(d). We observe that the only periodic solutions which we have found as the generalizations of dark vector solitons correspond to vector solitons with either zero or π phase change across the soliton.

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